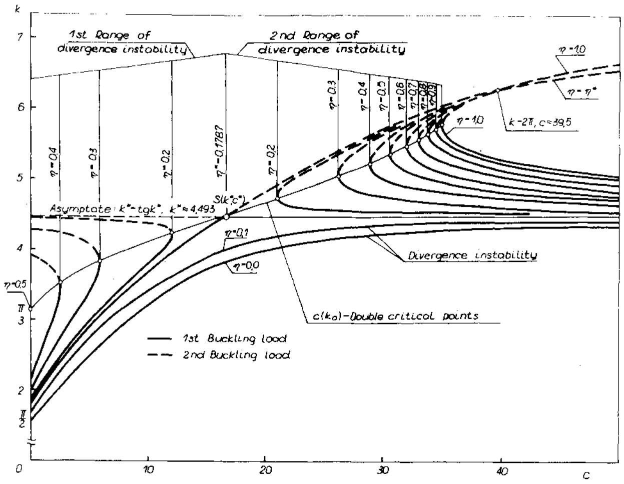


**Fig. 2 Regions of divergence instability for clamped, elastically supported column under force that follows according to parameter $\eta$.**

$$\eta^* = -\cos k^* / (1 - \cos k^*) \tag{8c}$$

From an analysis of Eq. (6a) and its general solution [Eq. (5)], one can easily find that the functions $c = c(k, \eta)$ have their extrema $c_0(\eta)$ at points $k = k_0$, with points $(k_0, c_0)$ belonging to curve $Q(k,c) = 0$ [with $P(k) \neq 0$]. The curves belonging to the family of Eq. (5) and curve $c(k_0)$ (double critical points) are shown in Fig. 2. The characteristics of the Eq. (5) curves are listed in Tables 1 and 2.

Considering the results obtained by Kounadis,[4] the region of divergence instability can be determined. If $c_0(\eta)$ is a maximum of Eq. (5), divergence instability occurs in the region $0 \leq c \leq c_0(\eta)$; if $c_0$ is a minimum, divergence instability takes place in the region $c_0(\eta) \leq c < \infty$. In the problem considered above, this occurs for values of the parameter $\eta > \eta^*$. In the case $\eta < \eta^*$, divergence instability occurs independently of the rigidity of a column support characterized by $c$ (cf. Fig. 1 and Table 2). The first buckling load for column at $\eta = \eta^*$ and $c > c^*$ is the equivalent of the critical force of a clamped, hinged column. For $\eta^* < \eta \leq 0.5$, divergence instability takes place within two ranges depending on the rigidity $c$ of a column support. On the other hand, for $0.5 < \eta \leq 1$, divergence instability occurs within one range for $c > c_0(\eta)$ (Fig. 2 and Table 2).

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# Efficient Repetitive Solution of Linear Equations with Varying Coefficients


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FINITE element analysis often deals with problems that involve the repetitive solution of a set of linear equations with both fixed and variable coefficients. Examples include the optimization of structures, electrical circuits, fluids models, etc. A method for the efficient solution of such sets of equations is developed and substantial reductions in effort are demonstrated. Savings can easily approach one or two orders of magnitude in the computational effort required. Equations are explicitly stated and operation count estimates are made.

This method is especially useful for equation sets containing a sufficient number of degrees of freedom to prohibit solutions on the user's digital computer because of time and cost constraints. The procedures suggested will make possible the solution to many problems that would otherwise require larger and more expensive computers. The mathematics are based upon an assumed parametric functional relationship in the coefficient matrix; however, the technique may be employed with other parametric relationships to derive similar results.

The linear equations are ordered as follows: 1) degrees of freedom with constant coefficients ($k$ set) and 2) degrees of freedom with variable coefficients ($v$ set). Variable terms in the coefficient matrix are removed prior to factoring and carried as auxiliary diagonal matrices. Factoring a matrix the size of the original coefficient matrix is required only once. Repetitive solutions require the factoring of a matrix equal in size to the number of degrees of freedom in the $v$ set. The operation count savings for matrix factoring is on the order of the ratio of the above matrix sizes cubed.

The linear system to be solved is assumed to have the form

$$MX = B \tag{1}$$

The partitioned matrix $M$ is defined by

$$M = \begin{bmatrix} A_{kk} & A_{kv} \\ A_{vk} & A_{vv} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \tag{2}$$

The $A$ matrix of Eq. (2) has constant coefficients only. Parametric variations are achieved by manipulation of the auxiliary diagonal $E$ and $D$ matrices. Substitution of Eq. (2) into Eq. (1) yields

$$\left[ \begin{bmatrix} A_{kk} & A_{kv} \\ A_{vk} & A_{vv} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \right] \begin{bmatrix} X_k \\ X_v \end{bmatrix} = \begin{bmatrix} B_k \\ B_v \end{bmatrix} \tag{3}$$

The $A$ matrix of Eq. (3) may now be expressed as the product of its lower and upper triangle of factors, $A = LU$. Expressing the factors in partition form and substituting into Eq. (3) yields

$$\begin{bmatrix} L_{kk} & 0 \\ L_{vk} & L_{vv} \end{bmatrix} \begin{bmatrix} U_{kk} & U_{kv}E \\ 0 & U_{vv}E \end{bmatrix} \begin{bmatrix} X_k \\ X_v \end{bmatrix} + \begin{bmatrix} 0 \\ DX_v \end{bmatrix} = \begin{bmatrix} B_k \\ B_v \end{bmatrix} \tag{4}$$

Received July 6, 1984; revision received Nov. 30, 1984. 

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Introducing the intermediate solution vector $Z$ defined by

$$\begin{bmatrix} Z_k \\ Z_v \end{bmatrix} = \begin{bmatrix} U_{kk} & U_{kv}E \\ 0 & U_{vv}E \end{bmatrix} \begin{bmatrix} X_k \\ X_v \end{bmatrix} \tag{5}$$

Eq. (4) is rewritten as

$$\begin{bmatrix} L_{kk} & 0 \\ L_{vk} & L_{vv} \end{bmatrix} \begin{bmatrix} Z_k \\ Z_v \end{bmatrix} + \begin{bmatrix} 0 \\ DX_v \end{bmatrix} = \begin{bmatrix} B_k \\ B_v \end{bmatrix} \tag{6}$$

The efficient repetitive solution of Eq. (1) is based on three relations inherent in Eqs. (5) and (6). Substituting the second matrix equation of Eq. (5) into the second matrix equation of Eq. (6) and rewriting the remaining equations yields

$$L_{kk}Z_k = B_k \tag{7}$$

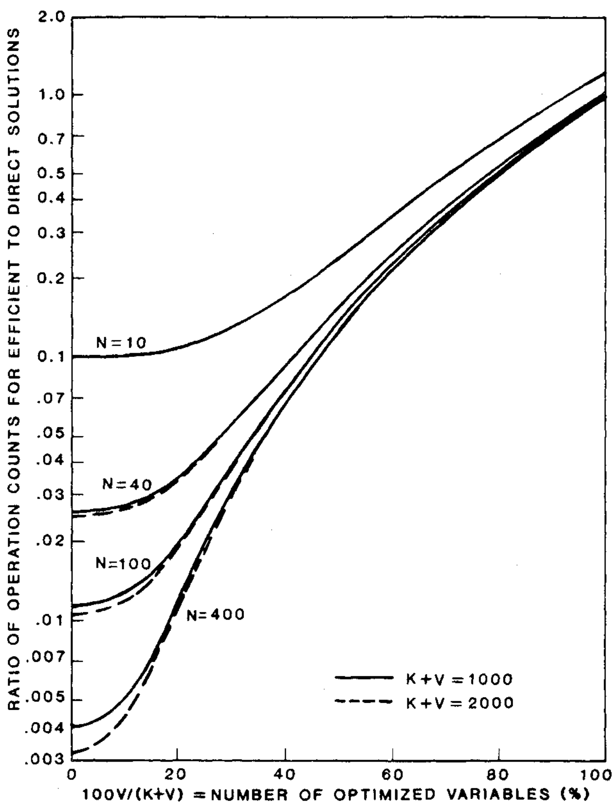


**Fig. 1 Ratio of operation count estimates for efficient to direct solutions.**

**Table 1 Direct solution operation count**

| Operation | Operation count |
|---|---|
| Factor $M$ | $N(K+V)^3/3$ |
| Backsolve for $X$ | $N(K+V)^2$ |

**Table 2 Efficient repetitive solution operation count**

| Operation | Operation count |
|---|---|
| Factor $A$ | $(K+V)^3/3$ |
| Backsolve $Z$, $k$ set | $K^2/2$ |
| Form $B-LZ$ | $KV+V$ |
| Multiply $LU$ | $V^3/3$ |
| Multiply $(LU)E+D$ | $NV^2$ |
| Solve for $X$, $v$ set | $N(V^3/3+V^2)$ |
| Solve for $X$, $k$ set | $N(K^2/2+KV+K+V)$ |

$$[L_{vv}U_{vv}E+D]X_v = B_v - L_{vk}Z_k \tag{8}$$

$$U_{kk}X_k = Z_k - U_{kv}EX_v \tag{9}$$

Equation (7) is solved for $Z_k$, Eq. (8) for $X_v$, and Eq. (9) for $X_k$. The solution for $Z_k$, the right-hand side of Eq. (8), and the $LU$ product of Eq. (8), are required only once since they do not vary with $E$ or $D$. The solution of Eqs. (8) and (9) must be obtained repetitively. The savings in computational effort result from factoring and backsolving Eq. (8), as opposed to Eq. (1).

Operation count estimates for the direct solution of a set of linear equations and the efficient repetitive solution presented herein are given in Tables 1 and 2. The count estimates are developed as a function of $K$, the number of degrees of freedom in the $k$ set; $V$, the number of degrees of freedom in the $v$ set; and $N$, the number of solutions to the system.

The ratio of operation counts for the efficient repetitive solution vs the direct solution of linear equations is presented in Fig. 1. These data were generated using the estimates of Tables 1 and 2 for a system of equations of order $K+V=1000$ and 2000. Data are presented for the $N=10$, 40, 100, and 400 solutions. Results for a system of equations of order 2000 are quite similar to the results of order 1000. Data for $K+V=3000$ have been evaluated and exhibit similar characteristics. Operation count ratios in the preceding problem size ranges are, practically speaking, independent of the size of the linear system.

The results of Fig. 1 may be interpreted as follows for the problem sizes considered. If the number of variable degrees of freedom in a linear system ($v$ set) is less than 10% of the total, the cost (in operation count) of all additional repetitive solutions is less than the cost of the original solution.

# The Role of Damping on Supersonic Panel Flutter

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## Introduction

ONE of the most important and interesting aspects of the theory of stability of elastic systems subjected to nonconservative forces is connected with the damping effects. In dealing with a double mathematical pendulum subjected on a free end to a follower (tangential) force, Ziegler[1] has found that the addition of a small amount of damping can reduce the value of the critical force in comparison with the value found without taking the damping into account. The effects of damping on the stability of elastic systems subjected to nonconservative forces have been studied by many authors. Some general results related to the stability of nonconservative systems have been obtained by Nemat-Nasser et al.[2,3] Bolotin and Zhinzher[4] conducted a systematic study of the damping effects on the stability of finite-degree-of-freedom and continuous systems subjected to nonconservative forces. A principal conclusion of this paper was that "for real laws of damping a considerable part of quasistability region belongs to the instability region. From this rigorous point of view the major-

Received Sept. 24, 1984; revision received Nov. 26, 1984.

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